XXXIII. *On the figure of the earth.* *By* GEORGE BIDDELL AIRY, *M. A. Fellow of Trinity College, Cambridge. Communicated by* J. F. W. HERSCHEL, *Esq. Sec. R. S.*

Read June 15, 1826.

THE ellipticity of the earth, deduced by Captain SABINE from a series of pendulum experiments the most extensive, and apparently the most deserving of confidence, that has ever been made, differs considerably from that which, as is generally believed, is indicated by geodetic measures. The difference can only be explained by errors of observation, by peculiarities of local circumstances, or by some defect in the theory which connects the figure of the earth with the variation of gravity on its surface: under the last head may be placed defects in the mathematical part of the theory, and errors in the assumptions of the original constitution and present state of the earth. It was with a view to ascertain the sufficiency of the mathematical theory, that I undertook the investigations contained in this paper. The celebrated proposition called CLAIRAUT'S theorem, by which the earth's ellipticity is inferred from the variation of gravity on its surface, is obtained only by the rejection of the squares and higher powers of the ellipticity. It is by the same rejection that the figure of the earth, supposed a heterogeneous fluid, is proved to be an elliptic spheroid. It appeared therefore probable, that a more accurate theory might introduce some modification into CLAIRAUT'S theorem, and might also show the figure of the earth to differ from an ellipsoid; and there was no reason to think that the first approximation to that figure was more accurate, than the first approximation to the motion of the moon's perigee. The result of my investigation does not at all serve to reconcile the pendulum observations made by Captain SABINE with the measures of degrees: and with respect to one object, which I hoped to obtain, I am therefore completely unsuccessful. The theory shows, however, that the earth's figure, on the usual suppositions as to its constitution, is not an elliptic spheroid; and the formulæ which I have obtained will give the means of determining very exactly the figure of the earth, when the experiments on the variation of gravity, or the measures of arcs on the earth's surface, shall be thought sufficiently accurate. As the subject is one whose interest is not confined to the present time, I have ventured to offer my investigations to the Royal Society.

The first part of the following sheets contains the theory of the heterogeneous earth, pushed so far as to include all the terms of the second order: it is succeeded by a comparison of this theory with Captain SABINE's results, and with the best arcs of the meridian that have been measured: and in the conclusion, I have offered some suggestions on the propriety of repeating some of these measures.

(1.) To ascertain the form of equilibrium of a fluid, it is first necessary to find the sum of the products of each particle by the reciprocal of its distance from any point of the fluid (LAPLACE, *Mecanique Celeste*, livre iii. n°. 23). This can be done only by assuming a function with indeterminate constants to represent the form of each stratum of equal density: then finding that sum, and applying to it the conditions of equilibrium, the values of the constants will be found. As we propose to carry our approximation to the second order of the difference from a sphere, or the second order of the ratio of the centrifugal force to the force of gravity, it is evident that, without something to guide us, this will be a work of considerable labour.

(2.) Here, however, we shall derive some assistance from former investigations. CLAIRAUT and LAPLACE have shown that, to the first order, the form of every surface of equal density is an elliptic spheroid: the difference, consequently, of any surface of equal density from an elliptic spheroid is only of the second order. If a be the polar semi-axis of an elliptic spheroid, $a(1+e)$ the equatorial semi-axis, μ' the sine of the latitude of any point, (the latitude being that which is usually termed the corrected latitude), R the radius drawn to that point, then $R = a\left(1 + e.\overline{1-\mu'^2} - \frac{3e^2}{2}\overline{\mu'^2-\mu'^4}\right)$. The radius then of a surface of equal density is $a\left(1 + e.\overline{1-\mu'^2} - \frac{3e^2}{2}.\overline{\mu'^2-\mu'^4}\right)$ + a quantity of the second order. Now, upon using the elliptic value of R, it would be found that the equation of equilibrium could not be satisfied, in consequence of the appearance of μ'^4: but no higher powers of μ' would enter into that equation. To enable us to take away these terms, R must be increased by a function of μ', containing none but the even powers of μ' as far as μ'^4. The most convenient form that we can take is $a.A(\mu'^4-\mu'^2)$, since it vanishes both at the pole and at the equator, and at middle latitudes expresses the depression of the surface below the ellipsoid whose axes are the same. The value of R then in the spheroidal surface of equal density is assumed to be $a\left(1+e\,.\,\overline{1-\mu'^2}-\left(\frac{3e^2}{2}+A\right).\,\overline{\mu'^2-\mu'^4}\right)$.

(3.) Let V be the sum of the products of each particle by the reciprocal of its distance from any point: let μ be the sine of the latitude of that point, r its distance from the centre: also let θ and ω be the latitude and longitude of that point, θ' and ω' those of any other point. Then it is easily found that their distance $=\sqrt{\{r^2-2rR(\sin\theta\,.\,\sin\theta'+\cos\theta\,.\,\cos\theta'\,.\,\cos\overline{\omega'-\omega})+R^2\}}=\sqrt{\{r^2-2rR(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\,.\,\cos\overline{\omega'-\omega})+R^2\}}$: let this $=z$. Suppose now the heterogeneous spheroid divided into wedges by planes passing through the axis, and suppose each of these wedges divided into pyramids whose vertices are at the centre: let $\delta\omega'$ be the angle of two planes, and $\delta\theta'$ the angle made by the two surfaces of a pyramid which cut these planes; and suppose the pyramid divided into frustra, the length of one being δR. Then the solid content of this frustrum is ultimately $R\,\delta\theta'\,.\,R\cos\theta'\delta\omega'\,.\,\delta R=R^2\,\delta R\,.\,\delta\omega'\,.\,\delta\mu'$ ultimately; and if ρ be its density, the product of the particle into the reciprocal of its distance from the given point is ultimately $\frac{\rho\,.\,R^2\,\delta R\,.\,\delta\omega'\,.\,\delta\mu'}{z}$. Consequently, to find the sum of those products for the spheroid, we must integrate $\frac{\rho R^2}{z}$ with respect to R, ω', and μ': or, which amounts to the same, we must integrate $\frac{\rho R^2}{z}\,.\,\frac{dR}{da}$ with respect to a, ω', and μ': that is, we must take $\int_a\int_{\omega'}\int_{\mu'}\frac{\rho R^2}{z}\,.\,\frac{dR}{da}$ *, or in

* I prefer this notation, as it does not necessarily carry with it the idea of infinitely small quantities.

the common notation $\int\int\int \frac{\rho R^2}{z} \cdot \frac{dR}{da} \cdot da \cdot d\omega' \cdot d\mu'$. The limits of μ' are -1 and $+1$: those of ω are o and 2π: and those of a are o, and the value of a at the external surface, which we shall call a.

(4.) To perform these integrations, it is necessary to expand the expression for z in a series of powers of $\frac{R}{r}$ or $\frac{r}{R}$: the former of course must be used for the strata interior to the point in question, and the latter for the strata exterior to that point, that the series may in both cases converge. Suppose then $\frac{1}{\sqrt{\left\{r^2 - 2rR\left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega'-\omega)\right) + R^2\right\}}} = \frac{Q^{(0)}}{r} + Q^{(1)} \cdot \frac{R}{r^2} + Q^{(2)} \cdot \frac{R^2}{r^3} + Q^{(3)} \cdot \frac{R^3}{r^4} +$ &c. for the former case, or $= \frac{Q^{(0)}}{R} + Q^{(1)} \cdot \frac{r}{R^2} + Q^{(2)} \cdot \frac{r^2}{R^3} + Q^{(3)} \cdot \frac{r^3}{R^4} +$ &c. for the latter, ($Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$, &c. being the same in both series): then $Q^{(i)}$ satisfies the following equation. (liv. iii. n°. 9)

$$o = \frac{d}{d\mu}\left\{\overline{1-\mu^2} \cdot \frac{dQ^{(i)}}{d\mu}\right\} + \frac{1}{1-\mu^2} \cdot \frac{d^2 Q^{(i)}}{d\omega^2} + i \cdot \overline{i+1} \cdot Q^{(i)}$$

which is likewise true if we put μ' for μ and ω' for ω. And the value of V now depends on the integral

$$\frac{1}{r^{i+1}}\int_a\int_{\omega'}\int_{\mu'} \rho \cdot Q^{(i)} \cdot R^{i+2} \cdot \frac{dR}{da} \text{ or } \frac{1}{r^{i+1}}\int_a \rho \int_{\omega'}\int_{\mu'} Q^{(i)} \cdot R^{i+2} \cdot \frac{dR}{da}$$

(since ρ is a function of a alone), for the interior strata, i being made successively $=$ o, 1, 2, 3, &c.; and on the integral $r^i\int_a \rho\int_{\omega'}\int_{\mu'} \frac{Q^{(i)}}{R^{i-1}} \cdot \frac{dR}{da}$, for the exterior strata, i being made $=$ o, 1, 2, 3, &c.

(5.) Suppose now R^{i+3} reduced into a series of the form

$B_0 Z'^{(0)} + B_, Z'^{(1)} + B_2 Z'^{(2)} +$ &c., where B_0, &c. are functions of a, and $Z'^{(k)}$ is a function of μ' that satisfies the equation

$$0 = \frac{d}{d\mu'}\left\{\overline{1-\mu'^2}.\frac{dZ'^{(k)}}{d\mu'}\right\} + k.\overline{k+1}.Z.^{(k)}$$

This equation is exactly similar to the equation above for $Q^{(i)}$, since R does not depend on $\omega' - \omega$, and therefore $\frac{d.^2 Z^{(k)}}{d\omega'^2}$ is $= 0$. Then $R^{i+2} \cdot \frac{dR}{da} = \frac{1}{i+3} \cdot \frac{d.R^{i+3}}{da} = \frac{1}{i+3} \cdot \frac{d}{da}\left(B_0 Z'^{(0)} + B_, Z'^{(1)} + B_2 Z'^{(2)} + \&c.\right)$, the general term of which is $\frac{1}{i+3} \cdot \frac{d.B_k}{da} . Z'^{(k)}$ The value of V then for the interior strata is the sum of all the quantities $\frac{1}{(i+3).r^{i+1}}\int_a \rho \frac{d.B_k}{da} \int_{\omega'}\int_{\mu'} Q^{(i)}. Z'^{(k)}$ upon giving to i and k the values 0, 1, 2, 3, &c.; and the integral with respect to a must be taken from $a = 0$ to $a =$ the value which it has in the stratum of equal density passing through the given point, which we shall call α.

(6.) For the strata exterior to the given point we must take the sum of the different values of $r^i\int_a \rho \int_{\omega'}\int_{\mu'}\frac{Q^{(i)}}{R^{i-1}} \cdot \frac{dR}{da}$. Now $\frac{1}{R^{i-1}} \cdot \frac{dR}{da} = \frac{1}{2-i} \cdot \frac{d}{da} \cdot \frac{1}{R^{i-2}}$, except $i = 2$, when $\frac{1}{R} \cdot \frac{dR}{da} = \frac{d}{da} . \log. R$. Suppose then $\frac{1}{R^{i-2}}$ or R^{2-i} expanded into the series $b_0 z'^{(0)} + b_, z'^{(1)} + b_2 z'^{(2)}$, &c. where $z'^{(k)}$ satisfies the same equation as before; then for the value of V we must take the sum of all the quantities $\frac{r^i}{2-i}\int_a \rho \frac{d.b_k}{da} \cdot \int_{\omega'}\int_{\mu'} Q^{(i)}.z'^{(k)}$, upon giving to k the values o, 1, 2, 3, &c. and to i the values o, 1, 3, 4, &c. For the value $i = 2$, we must suppose log. R expanded into the series $c_0 y'^{(0)} + c_1 y'^{(1)} + c_2 y'^{(2)} +$ &c. and must take the sum of the quantities $r^3 \int_a \rho \frac{d \cdot c_k}{d a} \int_{\omega'} \int_{\mu'} Q^{(2)} \cdot y'^{(k)}$, making k successively $=$ o, 1, 2, &c. The integrals with respect to a must be taken from $a = \alpha$ to $a =$ a.

(7.) Now LAPLACE has shown (*Mec. Cel.* liv. 3. n°. 12). that $\int_{\omega'} \int_{\mu'} Q^{(i)} . Z'^{(k)}$ from $\mu' = -1$ to $\mu' = +1$, and from $\omega' = 0$ to $\omega' = 2\pi$, will always be $=$ o, except $k = i$. And it appears also (liv. 3. n°. 11.) that $U^{(i)} = \frac{4 \alpha \pi}{2i+1} a^{i+3} . Y^{(i)}$; but $U^{(i)}$ (see n°. 17.) is there $= \alpha . a^{i+3} \int_{\omega'} \int_{\mu'} Q^{(i)} . Y'^{(i)}$ between the same limits, $Y^{(i)}$ being the value of $Y'^{(i)}$ when μ and ω are put for μ' and ω': hence $\int_{\omega'} \int_{\mu'} Q^{(i)} . Y'^{(i)} = \frac{4\pi}{2i+1} Y^{(i)}$, where $Y'^{(i)}$ is any function of μ' and ω' satisfying the equation $0 = \frac{d}{d\mu'} \left\{ \overline{1 - \mu'^2} \cdot \frac{d . Y'^{(i)}}{d\mu'} \right\} + \frac{1}{1 - \mu'^2} \cdot \frac{d^2 Y'^{(i)}}{d\omega'^2} + i . \overline{i+1} . Y'^{(i)}$.

Consequently $\int_{\omega'} \int_{\mu'} Q^{(i)} . Z'^{(i)} = \frac{4\pi}{2i+1} Z^{(i)} : \int_{\omega'} \int_{\mu'} Q^{(i)} z'^{(i)} = \frac{4\pi}{2i+1} z^{(i)}$; and $\int_{\omega'} \int_{\mu'} Q^{(2)} . y'^{(2)} = \frac{4\pi}{5} y^{(2)}$ where $Z^{(i)}$, $z^{(i)}$, and $y^{(2)}$ are the values of $Z'^{(i)}$, $z'^{(i)}$, and $y'^{(2)}$, when μ' is changed to μ.

(8.) Since we propose to include only the second order of e, and since to that order no powers of μ' beyond the fourth are found in R or any function of R, it follows, that $Q^{(i)} Z'^{(4)}$ and $Q^{(i)} z'^{(4)}$ will be the last terms to be integrated, and therefore i will not exceed 4. And as even powers only of μ' are found in R, there will be no terms as $Z'^{(i)}$ in which i is odd, and therefore it is not necessary to consider any but the even values of i. Now for the interior spheroids, when $i = 0$,

$$R^{i+3} = R^3 = a^3 \left\{ 1 + 3e.\overline{1-\mu'^2} + 3e^2.\overline{1-\mu'^2}|^2 - \left(3A + \frac{9e^2}{2}\right).\overline{\mu'^2 - \mu'^4} \right\}:$$

and since by LAPLACE's formula (liv. 3. n°. 16), $Z'^{(i)}$, when it does not depend on ω', is a multiple of $\mu'^i - \frac{i.\overline{i-1}}{2.\overline{2i-1}}.\mu'^{i-2} + \frac{i.\overline{i-1}.\overline{i-2}.\overline{i-3}}{2.4.\overline{2i-1}.\overline{2i-3}}\mu'^{i-4}$ &c. we must resolve R^3 into multiples of $\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}$, of $\mu'^2 - \frac{1}{3}$, and constants. Thus we have $R^3 =$

$$a^3\left(1 + 2e + e^2 - \frac{2}{5}A\right) - a^3\left(3e + \frac{57}{14}e^2 + \frac{3}{7}A\right)\left(\mu'^2 - \frac{1}{3}\right) + \left(\frac{15}{2}e^2 + 3A\right).\left(\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}\right)$$

$$\text{whence } B_0 = a^3\left(1 + 2e + e^2 - \frac{2}{5}A\right),\ Z^{(0)} = 1.$$

Similarly when $i = 2$, $R^5 =$

$$\text{a constant} - a^5\left(5e + \frac{25}{2}e^2 + \frac{5}{7}A\right)\left(\mu'^2 - \frac{1}{3}\right) + \text{a multiple of } \left(\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}\right):$$

$$\text{whence } B_2 = -a^5\left(5e + \frac{25}{2}e^2 + \frac{5}{7}A\right),\ Z^{(2)} = \mu^2 - \frac{1}{3}.$$

And when $i = 4$, $R^7 =$

$$\text{a constant} + \text{a multiple of } \overline{\mu'^2 - \tfrac{1}{3}} + a^7\left(\frac{63}{2}e^2 + 7A\right).\left(\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}\right):$$

$$\text{whence } B_4 = a^7\left(\frac{63}{2}e^2 + 7A\right),\ Z^{(4)} = \mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}.$$

(9.) Collecting then the different terms of $\frac{1}{(i+3)r^{i+1}}\int_a \rho \frac{dB_k}{da}\int_{\omega'}\int_{\mu'} Q^{(i)}.Z'^{(k)}$, which by (7) has no value except $k = i$, in which case $\int_{\omega'}\int_{\mu'} Q^{(i)}.Z'^{(k)} = \frac{4\pi}{2i+1}Z^{(i)}$, we have V for the interior strata $=\frac{4\pi}{1.3.r}\int_a \rho \frac{d.a^3\left(1+2e+e^2-\frac{2}{5}A\right)}{da}$

$-\frac{4\pi}{5.5.r^3}\int_a \rho . \frac{d.a^5\left(5e+\frac{25}{2}e^2+\frac{5}{7}A\right)}{da}\left(\mu^2-\frac{1}{3}\right)+\frac{4\pi}{9.7.r^5}\int_a \rho \frac{d.a^7\left(\frac{63}{2}e^2+7A\right)}{da}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right).$

Let $\int_a \rho \frac{d.a^3(1+2e+e^2-\frac{2}{5}A)}{da}=\phi(a)$; $\int_a \rho \frac{d.a^5(e+\frac{5}{2}e^2+\frac{1}{7}A)}{da}=$ $\psi(a)$; $\int_a \rho \frac{d.a^7(e^2+\frac{2}{9}A)}{da}=v(a)$, the integrals being made to vanish when $a=0$: taking them to $a=\alpha$, V for the interior strata $=\frac{4\pi}{3}\left\{\frac{\phi(\alpha)}{r}-\frac{3}{5}.\frac{\psi(\alpha)}{r^3}.\overline{\mu^2-\frac{1}{3}}+\frac{3}{2}.\frac{v(\alpha)}{r^5}.\overline{\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}}\right\}$

(10). For the strata exterior to the point in question, when $i=0, R^{2-i}=R^2$

$=a^2(1+2e.\overline{1-\mu'^2}+e^2.\overline{1-\mu'^2}|^2-(3e^2+2A).\overline{\mu'^2-\mu'^4})$

$=a^2(1+\frac{4e}{3}+\frac{2e^2}{15}-\frac{4}{15}A)-a^2(2e+\frac{11}{7}e^2+\frac{2}{7}A).\overline{\mu'^2-\frac{1}{3}}$

$+a^2(4e^2+2A).\overline{\mu'^4-\frac{6}{7}\mu'^2+\frac{3}{35}}$;

whence $b_0=a^2(1+\frac{4e}{3}+\frac{2e^2}{15}-\frac{4}{15}A)$, $z^{(0)}=1$.

When $i=2$, log R

$=\log a+e.\overline{1-\mu'^2}-\frac{e^2}{2}.\overline{1-\mu'^2}|^2-\left(\frac{3e^2}{2}+A\right).\overline{\mu'^2-\mu'^4}$

$=$ a constant term $-\left(e-\frac{5}{14}e^2+\frac{A}{7}\right).\overline{\mu'^2-\frac{1}{3}}$

$+$ a multiple of $\overline{\mu'^4-\frac{6}{7}\mu'^2+\frac{3}{35}}$;

whence $c_2=-\left(e-\frac{5}{14}e^2+\frac{A}{7}\right)$, $y^{(2)}=\mu^2-\frac{1}{3}$.

When $i=4$, $R^{2-i}=\frac{1}{R^2}$

$=\frac{1}{a^2}\left(1-2e.\overline{1-\mu'^2}+3e^2.\overline{1-\mu'^2}|^2+(3e^2+2A).\overline{\mu'^2-\mu'^4}\right)$

$=$ a constant $+$ a multiple of $\overline{\mu'^2-\frac{1}{3}}-\frac{2A}{a^2}.\overline{\mu'^4-\frac{6}{7}\mu'^2+\frac{3}{35}}$

whence $b_4=-\frac{2A}{a^2}$, $z^{(4)}=\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}$.

(11). Collecting the terms now in the same manner as in (9), according to the directions in (6), and observing the values of the integrals in (7), we find that part of V which depends on the exterior strata $= \frac{4\pi}{2.1}\int_a \rho \frac{d.a^2(1+\frac{4e}{3}+\frac{2e^2}{15}-\frac{4}{15}A)}{da} - \frac{4\pi}{5} r^2 \int_a \rho \frac{d(e-\frac{5}{14}e^2+\frac{A}{7})}{da} \cdot \overline{\mu^2-\frac{1}{3}} + \frac{4\pi}{2.9} r^4 \int_a \rho \cdot \frac{d\frac{A}{a^2}}{da} \cdot \overline{\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}}$.

Let $\int_a \rho \frac{d.a^2(1+\frac{4e}{3}+\frac{2e^2}{15}-\frac{4}{15}A)}{da} = \tau(a)$; $\int_a \rho \frac{d(e-\frac{5}{14}e^2+\frac{A}{7})}{da} = \chi(a)$; $\int_a \rho \cdot \frac{d}{da} \cdot \frac{A}{a^2} = \sigma(a)$; then taking the integrals from $a = \alpha$ to $a = \mathrm{a}$ we have for the exterior strata,

$$V = \frac{4\pi}{3}\left\{\frac{3}{2}\left\{\tau(\mathrm{a})-\tau(\alpha)\right\} - \frac{3}{5}r^2\left\{\chi(\mathrm{a})-\chi(\alpha)\right\}\overline{\mu^2-\frac{1}{3}} + \frac{r^4}{3}\left\{\sigma(\mathrm{a})-\sigma(\alpha)\right\}\overline{\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}}\right\}.$$

(12). Adding together the expressions in (9) and (11,) we find for the complete value of V, $\frac{4\pi}{3}\left\{\frac{\varphi(\alpha)}{r} + \frac{3}{2}\left\{\tau(\mathrm{a})-\tau(\alpha)\right\} + \left(\frac{\psi(\alpha)}{r^3} + r^2\left\{\chi(\mathrm{a})-\chi(\alpha)\right\}\right) \cdot \frac{1-3\mu^2}{5} + \left(\frac{v(\alpha)}{r^5} + \frac{2}{9}r^4\left\{\sigma(\mathrm{a})-\sigma(\alpha)\right\}\right) \cdot \left(\frac{9}{70} - \frac{9}{7}\mu^2 + \frac{3}{2}\mu^4\right)\right\}$

(13.) When a fluid is in equilibrium,* if x, y, and z, be the co-ordinates of a point in it, and P, Q, R, the forces in those directions, then at a surface of equal density $P\delta x + Q\delta y + R\delta z = 0$ (liv. i. n°. 17); and the equilibrium is impossible except $P\delta x + Q\delta y + R\delta z$ be the complete variation of some function U. The equation then to a surface of equal density is $U = C$. In the present instance,

* I have not considered the second condition of equilibrium, given by Mr. IVORY in the Philosophical Transactions for 1824, as the reasoning upon which that Gentleman has founded the necessity of such a condition, appears to me altogether defective.

if the ordinates be measured from the centre, and x be the axis of revolution, and T the time of revolution, the only force besides the attraction is the centrifugal force; the resolved part of which in the direction of x is o, that in the direction of y is $\frac{4\pi^2}{T^2} y$, that in the direction of z is $\frac{4\pi^2}{T^2} z$. This contributes to δU the terms $\frac{4\pi^2}{T^2}(y\,\delta y + z\,\delta z)$, and to U the terms $\frac{2\pi^2}{T^2}(y^2 + z^2) = \frac{4\pi}{3} \cdot \frac{3\pi}{2T^2} r^2 \cdot \overline{1-\mu^2}$. And if δM be any attracting particle whose co-ordinates are x', y', z', its attractions in those directions are

$$\frac{(x'-x)\,\delta M}{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)^{\frac{3}{2}}},\quad \frac{(y'-y)\,\delta M}{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)^{\frac{3}{2}}},$$

$\frac{(z'-z)\,\delta M}{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)^{\frac{3}{2}}}$: and consequently it contributes to δU the term $\delta M \frac{(x'-x)\,\delta x + (y'-y)\,\delta y + (z'-z)\,\delta z}{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)^{\frac{3}{2}}}$, and to U the term $\frac{\delta M}{\sqrt{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)}}$. The expression then which the attraction of the whole adds to U is the sum of all the quantities $\frac{\delta M}{\sqrt{\left(\overline{x'-x}|^2 + \overline{y'-y}|^2 + \overline{z'-z}|^2\right)}}$, or the sum of the products of each particle by the reciprocal of its distance from the attracted point; that is, V. The whole value of U then is $V + \frac{4\pi}{3} \cdot \frac{3\pi}{2T^2} \cdot r^2 \cdot \overline{1-\mu^2}$; and the equation to a surface of equal density is $C = \frac{4\pi}{3}\Big\{\frac{\phi(\alpha)}{r} + \frac{3}{2}\{\tau(a) - \tau(\alpha)\}$

$$+ \left(\frac{\psi(\alpha)}{r^3} + r^2\{\chi(a) - \chi(\alpha)\}\right) \cdot \frac{1-3\mu^2}{5} + \frac{3\pi}{2T^2} r^2 \cdot \overline{1-\mu^2}$$

$$+ \left(\frac{v(\alpha)}{r^5} + \frac{2}{9} r^4 \{\sigma(a) - \sigma(\alpha)\}\right) \cdot \left(\frac{9}{70} - \frac{9}{7}\mu^2 + \frac{3}{2}\mu^4\right)\Big\}$$

(14.) If then for r we put its value in terms of the semi-axis of the spheroid which terminates it, and the angle which it makes with the equator, we shall have an equation which, so far as it depends on that angle, must be identically true.

Now if ε and E are the same functions of α which e and A are of a, $\frac{1}{r}$ to the second order $= \frac{1}{\alpha}\left(1 - \varepsilon \cdot \overline{1-\mu^2} + \varepsilon^2 \cdot \overline{1 - \frac{\mu^2}{2} - \frac{\mu^4}{2}} + \text{E} \cdot \overline{\mu^2 - \mu^4}\right)$: $\frac{1}{r^3}$ to the first order $= \frac{1}{\alpha^3}$ $(1 - 3\varepsilon \cdot \overline{1-\mu^2})$; $r^2 = \alpha^2(1 + 2\varepsilon \cdot \overline{1-\mu^2})$; $\frac{1}{r^5}$ including no small quantities $= \frac{1}{\alpha^5}$; $r^4 = \alpha^4$. These are not to be taken farther, because $\psi(\alpha)$, $\chi(\alpha)$, and $\frac{3\pi}{2\text{T}^2}$, are of the first order, and $\upsilon(\alpha)$ and $\sigma(\alpha)$ of the second order. Substituting these values in the last equation, we find C $=$

$$\frac{4\pi}{3}\left\{\begin{aligned}
&\frac{\varphi(\alpha)}{\alpha} + \frac{3}{2}\{\tau(\text{a}) - \tau(\alpha)\} \\
&- \frac{\varepsilon\varphi(\alpha)}{\alpha} \cdot \overline{1-\mu^2} + \left(\frac{\psi(\alpha)}{\alpha^3} + \alpha^2\{\chi(\text{a}) - \chi(\alpha)\}\right) \cdot \frac{1-3\mu^2}{5} + \frac{3\pi}{2\text{T}^2}\alpha^2 \cdot \overline{1-\mu^2} \\
&+ \frac{\varepsilon^2 \cdot \varphi(\alpha)}{\alpha} \cdot \overline{1 - \frac{\mu^2}{2} - \frac{\mu^4}{2}} + \frac{\text{E}\varphi(\alpha)}{\alpha}\overline{\mu^2 - \mu^4} - \frac{3\varepsilon\psi(\alpha)}{\alpha^3} \cdot \overline{1-\mu^2} \cdot \frac{1-3\mu^2}{5} + \frac{3\pi}{2\text{T}^2} 2\varepsilon\alpha^2 \cdot \overline{1-\mu^2}^2 \\
&+ 2\varepsilon\alpha^2\{\chi(\text{a}) - \chi(\alpha)\}\,\overline{1-\mu^2} \cdot \frac{1-3\mu^2}{5} + \left(\frac{\upsilon(\alpha)}{\alpha^5} + \frac{2}{9}\alpha^4\{\sigma(\text{a}) - \sigma(\alpha)\}\right) \cdot \left(\frac{9}{70} - \frac{9}{7}\mu^2 + \frac{3}{2}\mu^4\right)
\end{aligned}\right.$$

Making the coefficient of $\mu^2 = 0$, and that of $\mu^4 = 0$, and observing that, as the equations which we shall find are general for all values of α, we may put a for α,

$$0 = \left\{\begin{aligned}
&\frac{e\varphi(a)}{a} - \frac{3}{5} \cdot \frac{\psi(a)}{a^3} - \frac{3}{5}a^2\{\chi(\text{a}) - \chi(a)\} - \frac{3\pi}{2\text{T}^2}a^2 \\
&- \frac{e^2\varphi(a)}{2a} + \frac{A \cdot \varphi(a)}{a} + \frac{12}{5} \cdot \frac{e\psi(a)}{a^3} - \frac{8}{5}ea^2\{\chi(\text{a}) - \chi(a)\} - \frac{3\pi}{2\text{T}^2}4ea^2 \\
&\qquad - \frac{9}{7}\left(\frac{\upsilon(a)}{a^5} + \frac{2}{9}a^4\{\sigma(\text{a}) - \sigma(a)\}\right)
\end{aligned}\right.$$

$$0 = \left\{\begin{aligned}
&- \frac{e^2\varphi(a)}{2a} - \frac{A\varphi(a)}{a} - \frac{9}{5} \cdot \frac{e\psi(a)}{a^3} + \frac{6}{5}ea^2\{\chi(\text{a}) - \chi(a)\} + \frac{3\pi}{2\text{T}^2}2ea^2 \\
&\qquad + \frac{3}{2}\left(\frac{\upsilon(a)}{a^5} + \frac{2}{9}a^4\{\sigma(\text{a}) - \sigma(a)\}\right)
\end{aligned}\right.$$

These are the equations of equilibrium; and since by differentiation they may be reduced to two differential equations, from which the two quantities A and e are to be found, their solution is possible, and the equilibrium with the assumed figure is therefore possible.

(15.) At the surface these equations become

$$0=\left(\mathrm{e}-\frac{\mathrm{e}^2}{2}+\mathrm{A}\right)\frac{\phi(\mathrm{a})}{\mathrm{a}}-\left(\frac{3}{5}-\frac{12}{5}\mathrm{e}\right).\frac{\psi(\mathrm{a})}{\mathrm{a}^3}-\frac{3\pi}{\mathrm{T}^2}\mathrm{a}^2(1+4\mathrm{e})-\frac{9}{7}.\frac{\upsilon(\mathrm{a})}{\mathrm{a}^5}.$$

$$0=-\left(\frac{\mathrm{e}^2}{2}+\mathrm{A}\right)\frac{\phi(\mathrm{a})}{\mathrm{a}}-\frac{9}{5}\mathrm{e}\frac{\psi(\mathrm{a})}{\mathrm{a}^3}+\frac{3\pi}{2\mathrm{T}^2}2\mathrm{e}\mathrm{a}^2+\frac{3}{2}.\frac{\upsilon(\mathrm{a})}{\mathrm{a}^5}.$$

(16). The force on any point in the direction of r is found by differentiating the expression at the end of (13) with respect to r. For since, by (13), the force in the direction of $x=\frac{d\mathrm{U}}{dx}$, if we conceive x to coincide with r, the force in the direction of $r=\frac{d\mathrm{U}}{dr}$. In this expression a is a function of r; but it will be found (as may also be proved by independent reasoning) that the terms produced by differentiation with respect to a destroy each other. The force then in the direction of r (with its sign changed, as we have to estimate not a repulsive but an attractive force) is

$$\frac{4\pi}{3}\left\{\frac{\phi(a)}{r^2}-\left(-\frac{3\psi(a)}{r^4}+2r\left\{\chi(\mathrm{a})-\chi(a)\right\}\right).\frac{1-3\mu^2}{5}-\frac{3\pi}{2\mathrm{T}^2}.2r.\overline{1-\mu^2}\right.$$
$$\left.-\left(-\frac{5\upsilon(a)}{r^6}+\frac{8}{9}r^3\left\{\sigma(\mathrm{a})-\sigma(a)\right\}\right).\left(\frac{9}{70}-\frac{9}{7}\mu^2+\frac{3}{2}\mu^4\right)\right\}.$$

At the external surface this is $\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{a})}{r^2}+\frac{3}{5}.\frac{\psi(\mathrm{a})}{r^4}.\overline{1-3\mu^2}-\frac{3\pi}{2\mathrm{T}^2}.2r.\overline{1-\mu^2}+\frac{\upsilon(\mathrm{a})}{r^6}\left(\frac{9}{14}-\frac{45}{7}\mu^2+\frac{15}{2}\mu^4\right)\right\}$. Observing that $\frac{1}{r^2}=\frac{1}{\mathrm{a}^2}\left(1-2\mathrm{e}.\overline{1-\mu^2}+3\mathrm{e}^2\,\overline{1-\mu^2}^2+2\mathrm{A}.\overline{\mu^2-\mu^4}\right)$; $\frac{1}{r^4}=\frac{1}{\mathrm{a}^4}\left(1-4\mathrm{e}.\overline{1-\mu^2}\right)$: $r=\mathrm{a}\left(1+\mathrm{e}.\overline{1-\mu^2}\right)$: $\frac{1}{r^6}=\frac{1}{\mathrm{a}^6}$; the force on a point of the surface in the direction of

$$r=\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{a})}{\mathrm{a}^2}-\frac{2\mathrm{e}.\phi(\mathrm{a})}{\mathrm{a}^2}.\overline{1-\mu^2}+\frac{3}{5}.\frac{\psi(\mathrm{a})}{\mathrm{a}^4}\,\overline{1-3\mu^2}-\right.$$
$$\frac{3\pi}{2\mathrm{T}^2}.2\mathrm{a}.\overline{1-\mu^2}+\frac{3\mathrm{e}^2.\phi(\mathrm{a})}{\mathrm{a}^2}.\overline{1-\mu^2}^2+\frac{\mathrm{A}2.\phi(\mathrm{a})}{\mathrm{a}^2}.\overline{\mu^2-\mu^4}$$
$$-\frac{12}{5}.\frac{\mathrm{e}.\psi(\mathrm{a})}{\mathrm{a}^4}.\overline{1-\mu^2}.\overline{1-3\mu^2}-\frac{3\pi}{2\mathrm{T}^2}.2\mathrm{e}\mathrm{a}.\overline{1-\mu^2}^2+\frac{\upsilon(\mathrm{a})}{\mathrm{a}^6}\left(\frac{9}{14}-\right.$$
$$\left.\left.\frac{45}{7}\mu^2+\frac{15}{2}\mu^4\right)\right\}.$$

(17.) Now since this is the resolved part of the whole gravity at any point, which is necessarily perpendicular to the surface, we shall find the whole gravity by dividing this expression by the cosine of the angle of the vertical. To the order which we are considering, the angle of the vertical is the same in the spheroid and in the ellipsoid with the same axes, and it is therefore $= 2\,e$. sin lat. cos lat. $= 2\,e\,.\,\mu\sqrt{1-\mu^2}$. Its cosine therefore $= 1 - 2\,e^2\,.\,\overline{\mu^2 - \mu^4}$: consequently gravity at any point $= \frac{4\pi}{3}\Big\{\frac{\varphi(a)}{(a)^2} - \frac{2\,e\,.\,\varphi(a)}{a^2}\,.\,\overline{1-\mu^2} + \frac{3}{5}\,.\,\frac{\psi(a)}{a^4}\,.\,\overline{1-3\,\mu^2} - \frac{3\,\pi}{2\,T^2}\,.\,2\,a\,.\,\overline{1-\mu^2} + \frac{2\,e^2\,.\,\varphi(a)}{a^2}\,.\,\overline{\mu^2-\mu^4} + \frac{3\,e^2\,.\,\varphi(a)}{a^2}\,.\,\overline{1-\mu^2} + \frac{2\,A\,\varphi(a)}{a^2}\,.\,\overline{\mu^2-\mu^4} - \frac{12}{5}\,.\,\frac{e\,.\,\psi(a)}{a^4}\,.\,\overline{1-\mu^2}\,.\,\overline{1-3\mu^2} - \frac{3\,\pi}{2\,T^2}\,.\,2\,e\,a\,.\,\overline{1-\mu^2}^{\,2} + \frac{\upsilon(a)}{a^6}\left(\frac{9}{14} - \frac{45}{7}\mu^2 + \frac{15}{2}\mu^4\right)\Big\}$. But this is in terms of μ, the sine of the corrected latitude: it will be more convenient to have it in terms of λ, the sine of the real latitude. Since the corr. lat. = real lat. $- 2\,e$. sin. lat. cos. lat. it is easily found that $\mu^2 = \lambda^2 - 4\,e\,.\,\overline{\lambda^2 - \lambda^4}$: substituting, gravity $= \frac{4\pi}{3}\Big\{\frac{\varphi(a)}{a^2}(1 - 2\,e + 3\,e^2) + \frac{\psi(a)}{a^4}\left(\frac{3}{5} - \frac{12}{5}\,e\right) - \frac{3\,\pi}{2\,T^2}\,a\,(2 + 2\,e) + \frac{9}{14}\,.\,\frac{\upsilon(a)}{a^6} + \left(\frac{\varphi(a)}{a^2}(2\,e - 3\,e^2) - \frac{\psi(a)}{a^4}\left(\frac{9}{5} - \frac{12}{5}\,e\right) + \frac{3\,\pi}{2\,T^2}\,a\,(2 + 2\,e) + \frac{15}{4}\,.\,\frac{\upsilon(a)}{a^6}\right)\lambda^2 - \left(\frac{\varphi(a)}{a^2}(6\,e^2 - 2\,A) - \frac{72}{5}\,.\,\frac{e\,.\,\psi(a)}{a^4} + \frac{3\,\pi}{2\,T^2}\,6\,a\,e + \frac{15}{2}\,.\,\frac{\upsilon(a)}{a^6}\right).\,\lambda^2.\,\overline{1-\lambda^2}\Big\}$.

(18.) From the equations of (15), $\frac{\psi(a)}{a^4} = \frac{\varphi(a)}{a^2}\left(\frac{5\,e}{3} + \frac{5\,e^2}{6} + \frac{5\,A}{21}\right) - \frac{3\,\pi}{2\,T^2}\,a\left(\frac{5}{3} + \frac{130}{21}\,e\right)$; $\frac{\upsilon(a)}{a^6} = \frac{\varphi(a)}{a^2}\left(\frac{7\,e^2}{3} + \frac{2\,A}{3}\right) - \frac{3\,\pi}{2\,T^2}\,.\,\frac{10}{3}\,a\,e$. Substituting these, gravity

$$= \frac{4\pi}{3}\Big\{\frac{\varphi(a)}{a^2}\left(1 - e + e^2 + \frac{4\,A}{7}\right) - \frac{3\,\pi}{2\,T^2}\,a\left(3 + \frac{27}{7}\,e\right)$$
$$+ \left(\frac{\varphi(a)}{a^2}\left(-\,e + 2\,e^2 + \frac{2}{7}\,A\right) + \frac{3\,\pi}{2\,T^2}\,a\left(5 + \frac{39}{7}\,e\right)\right).\,\lambda^2$$
$$- \left(\frac{\varphi(a)}{a^2}\left(-\frac{e^2}{2} + 3\,A\right) + \frac{3\,\pi}{2\,T^2}\,5\,a\,e\right).\,\lambda^2.\,\overline{1-\lambda^2}\Big\}.$$

(19.) Let m be the ratio of the centrifugal force at the equator to gravity at the equator. The centrifugal force there $= \frac{4\pi^2}{T^2} a(1+e) = \frac{4\pi}{3} \cdot \frac{3\pi}{2T^2} 2a(1+e)$: the gravity there $= \frac{4\pi}{3}\left\{\frac{\varphi(a)}{a^2}(1-e) - \frac{3\pi}{2T^2} \cdot 3a\right\}$: therefore $\frac{3\pi}{2T^2} a(2+2e) = \frac{\varphi(a)}{a^2}(m - me) - \frac{3\pi}{2T^2} a \cdot 3m$; whence $\frac{3\pi}{2T^2} a = \frac{\varphi(a)}{a^2} \cdot \frac{m - me}{2+2e+3m} = \frac{\varphi(a)}{a^2}\left(\frac{m}{2} - me - \frac{3m^2}{4}\right)$. Making this substitution, gravity $= \frac{4\pi}{3} \cdot \frac{\varphi(a)}{a^2}\left\{\left(1 - e - \frac{3}{2}m + e^2 + \frac{15}{14}me + \frac{9}{4}m^2 + \frac{4}{7}A\right) + \left(\frac{5m}{2} - e + 2e^2 - \frac{31}{14}me - \frac{15}{4}m^2 + \frac{2}{7}A\right)\lambda^2 - \left(\frac{5}{2}me - \frac{e^2}{2} + 3A\right) \cdot \lambda^2 \cdot \overline{1-\lambda^2}\right\}$
$= \frac{4\pi}{3} \cdot \frac{\varphi(a)}{a^2}\left(1 - e - \frac{3m}{2} + e^2 + \frac{15}{14}me + \frac{9}{4}m^2 + \frac{4}{7}A\right) \times \left\{1 + \left(\frac{5m}{2} - e + e^2 - \frac{17}{14}me + \frac{2}{7}A\right) \cdot \lambda^2 - \left(\frac{5}{2}me - \frac{e^2}{2} + 3A\right) \cdot \lambda^2 \cdot \overline{1-\lambda^2}\right\}$. If then the equatorial gravity be represented by G, the gravity in latitude l will be

$$G\left\{1 + \left(\frac{5m}{2} - e + e^2 - \frac{17}{14}me + \frac{2}{7}A\right)\sin.^2 l - \left(\frac{5}{2}me - \frac{e^2}{2} + 3A\right)\sin^2 l \cdot \cos^2 l\right\}$$

Whatever therefore be the law of density in the interior of the earth, the gravity at any point of the surface can be expressed from a knowledge of the form of that surface only. This is an extension of CLAIRAUT'S theorem.

(20). We shall now proceed to find an expression for the length of an arc of the meridian included between two given latitudes. If θ be the corrected latitude, and $u = \frac{1}{r}$, the radius of curvature is $\frac{\left\{1 + \frac{1}{u^2} \cdot \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{u + \frac{d^2 u}{d\theta^2}}$. Here $u = \frac{1}{a}(1 - e.\overline{1-\mu^2} + e^2 . \overline{1 - \frac{\mu^2}{2} - \frac{\mu^4}{2}} + A . \overline{\mu^2 - \mu^4}) = \frac{1}{a}(1 - \frac{e}{a} + \frac{9e^2}{16} + \frac{A}{8} -$

$\left(\frac{e}{2} - \frac{e^2}{2}\right) \cos 2\,\theta - \left(\frac{e^2}{16} + \frac{A}{8}\right) \cos 4\,\theta \Big\}$, and the radius of curvature $= a\left\{1 + \frac{e}{2} + \frac{25\,e^2}{16} - \frac{A}{8} - \frac{3\,e}{2} \cos 2\,\theta - \left(\frac{9\,e^2}{16} + \frac{15}{8}A\right) \cos 4\,\theta\right\}$. Let this $= \gamma$. It was found that $\mu^2 = \lambda^2 - 4\,e\,.\,\overline{\lambda^2 - \lambda^4}$, or $\sin^2\theta = \sin^2 l - 4\,e\,.\,\sin^2 l\,.\,\cos^2 l$; from which $\cos 2\,\theta = \cos 2\,l + e - e\cos 4\,l$: hence $\gamma = a\left\{1 + \frac{e}{2} + \frac{e^2}{16} - \frac{A}{8} - \frac{3\,e}{2} \cos 2\,l + \left(\frac{15\,e^2}{16} - \frac{15}{8}A\right) \cos 4\,l\right\}$. Now since l is the angle made by the normal with a fixed line, the increment of the arc, corresponding to the increment δl of the latitude, is ultimately $\gamma\,\delta l$, and the arc $= \int_l \gamma$; integrating therefore from $l = L$ to $l = L'$, and making $a\left(1 + \frac{e}{2} + \frac{e^2}{16} - \frac{A}{8}\right) = R$, the arc included between the latitudes L and L$' =$

$$R\left\{L' - L - \left(\frac{3\,e}{4} - \frac{3\,e^2}{8}\right).(\sin 2\,L' - \sin 2\,L) + \left(\frac{15\,e^2}{64} - \frac{15\,A}{32}\right).(\sin 4\,L' - \sin 4\,L)\right\}.$$

(21.) An arc of a parallel in latitude l, comprehending the difference of longitude D, is most easily obtained by observing that the decrement of its radius upon increasing l is equal to $\sin l \times$ the increment of the arc $= \sin l \times \gamma\,\delta l$, and therefore the radius $= -\int_l \gamma \sin l$; the integral being corrected so as to become o when $l = 90°$. We must therefore integrate $-R\sin l\left\{1 - \left(\frac{3\,e}{2} - \frac{3\,e^2}{4}\right) \cos 2\,l + \left(\frac{15\,e^2}{16} - \frac{15\,A}{8}\right) \cos 4\,l\right\}$. This gives the radius of the parallel $= R\left\{\left(1 + \frac{3\,e}{4} - \frac{3\,e^2}{8}\right) \cos l - \left(\frac{e}{4} + \frac{e^2}{32} - \frac{5\,A}{16}\right) \cos 3\,l + \left(\frac{3\,e^2}{32} - \frac{3\,A}{16}\right) \cos 5\,l\right\}$; and therefore the arc corresponding to the difference of longitude

$$D = D.R. \left\{ \left(1 + \frac{3e}{4} - \frac{3e^2}{8}\right) \cos l - \left(\frac{e}{4} + \frac{e^2}{32} - \frac{5A}{16}\right) \cos 3l + \left(\frac{3}{32} e^2 - \frac{3A}{16}\right) \cos 5l \right\}.$$

I now proceed to compare this theory with observation. In selecting from the numerous determinations of the variation of gravity one set for our present purpose, there can be no hesitation in fixing on Captain Sabine's, as extending over the greatest arc of latitude, and as being made by the same observer with the same instruments at no great interval of time. The following are the places of his observations and their latitudes, with the lengths of the seconds' pendulum determined by him.

Place.	Latitude.	Length of seconds' pendulum in inches.
	° ′ ″	
St. Thomas .	24 41,2	39,02074
Maranham .	2 31 43,3	,01214
Ascension .	7 55 47,8	,02410
Sierra Leone .	8 29 27,9	,01997
Trinidad . .	10 38 56	,01884
Bahia . . .	12 59 21	,02425
Jamaica . .	17 56 7,6	,03510
New York .	40 42 43,2	,10168
London . .	51 31 8,4	,13929
Drontheim .	63 25 54,2	,17456
Hammerfest .	70 40 5,3	,19519
Greenland .	74 32 18,6	,20335
Spitzbergen .	79 49 57,8	,21469

The irregularities in the lengths at the places nearest to the equator are considerable; but the number of these places is so great, that the errors will probably destroy each other. Assuming the length of the seconds' pendulum $= M + N \cos 2 \text{ lat} + P \cos 4 \text{ lat}$. the errors are

$$-39{,}02074 + M + N \times {,}99989 + P \times {,}99959$$
$$-39{,}01214 + M + N \times {,}99610 + P \times {,}98445$$
$$-39{,}02410 + M + N \times {,}96194 + P \times {,}85063$$
$$-39{,}01997 + M + N \times {,}95640 + P \times {,}82938$$
$$-39{,}01884 + M + N \times {,}93171 + P \times {,}73611$$
$$-39{,}02425 + M + N \times {,}89895 + P \times {,}61626$$
$$-39{,}03510 + M + N \times {,}81034 + P \times {,}31330$$
$$-39{,}10168 + M + N \times {,}14910 - P \times {,}95552$$
$$-39{,}13929 + M - N \times {,}22559 - P \times {,}89821$$
$$-39{,}17456 + M - N \times {,}59992 - P \times {,}28024$$
$$-39{,}19519 + M - N \times {,}78082 + P \times {,}21936$$
$$-39{,}20335 + M - N \times {,}85784 + P \times {,}47184$$
$$-39{,}21469 + M - N \times {,}93767 + P \times {,}75849$$

To determine M, N, and P, by the method of least squares, we must form three equations, by making equal to nothing the sum of these errors—first, when each is multiplied by the coefficient of M in that line, then when each is multiplied by the coefficient of N, and finally when each is multiplied by the coefficient of P. Thus we find

$$0 = -508{,}18390 + M \times 13 + N \times 3{,}30259 + P \times 4{,}64544$$
$$0 = -128{,}29476 + M \times 3{,}30259 + N \times 8{,}82265 + P \times 4{,}02629$$
$$0 = -181{,}31213 + M \times 4{,}64544 + N \times 4{,}02629 + P \times 7{,}04391$$

By the solution of these equations $M = 39{,}11647$: $N = -{,}10146$; $P = {,}00106$; and the length of the seconds' pendulum in inches $= 39{,}11647 - {,}10146 \times \cos 2 \text{ lat} + {,}00106$

$\times$ cos 4 lat $=$ 39,01606 $+$,20292 $\times$ sin^2 lat $-$,00844 $\times$ sin^2 lat . cos^2 lat . $=$ 39,01606 $\left\{1 + ,005201 \times \sin^2 \text{lat} - ,000216 \times \sin^2 \text{lat} . \cos^2 \text{lat}\right\}$; and gravity is proportional to this. Comparing the coefficients of sin^2 lat and sin^2 lat . cos^2 lat with those in the expression of Art. 19, we have the following equations

$$\frac{5m}{2} - e + e^2 - \frac{17}{14} m e + \frac{2}{7} A = ,005201$$

$$\frac{5}{2} m e - \frac{e^2}{2} + 3 A \qquad = ,000216$$

From the length of the equatorial pendulum, supposing the equatorial radius of the earth $=$ 3486908 fathoms, we find $m = ,003464$. And in the terms of the second order instead of e we may put $\frac{1}{290}$, which is certainly not far from the truth. Making these substitutions, the equations become

$$,008657 - e + \frac{2}{7} A = ,005201$$

$$,000024 + 3 A \qquad = ,000216$$

From these, $e = ,003474 = \frac{1}{287,8}$; and $A = 000064$. The sign of A shows that the earth is less protuberant at the latitude of 45° than an ellipsoid of the same polar and equatorial radii. For the elevation of the spheroid above the ellipsoid $= a A . \mu^2 . \overline{\mu^2 - 1}$, where $\mu =$ sine of latitude; making the latitude $= 45°$, $\mu = \frac{1}{\sqrt{2}}$, and the elevation of the spheroid is $-\frac{A}{4}$: as A here is positive, this elevation is negative, or the spheroid at that latitude is depressed below the ellipsoid. This I should be inclined to expect: for though I have not been able to solve the differential equation in *A*, even in the cases in which the differential equation for *e* can be solved to the first order, yet by the examination of a simple hypothesis (that of a homogeneous fluid whose density is very small surrounding a central nucleus), it may easily be seen that in this case the spheroid is not so far protuberant in middle latitudes as the ellipsoid. If the attraction of the fluid be neglected, and the central mass be called m, the force on any point in the direction of $x = \frac{-mx}{(x^2+y^2+z^2)^{\frac{3}{2}}}$: that in the direction of $y = \frac{-my}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{4\pi^2}{T^2}y$; that in $z = \frac{-mz}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{4\pi^2}{T^2}z$. Hence $U = \frac{m}{\sqrt{(x^2+y^2+z^2)}} + \frac{2\pi^2}{T^2}\overline{y^2+z^2} = C$; or if $y^2+z^2 = v^2$, $\frac{m}{\sqrt{x^2+v^2}} + \frac{2\pi^2}{T^2}v^2 = C$; whence $x^2 = \frac{m^2}{\left(c - \frac{2\pi^2}{T^2}v^2\right)^2} - v^2 = \frac{m^2}{C^2} - \left(1 - \frac{4\pi^2 m^2}{C^3 T^2}\right)v^2 + \frac{6\pi^4 m^2}{C^4 T^4}v^4$ nearly, supposing $\frac{2\pi^2}{CT^2}$ small; and this may be put under the form $(a^2 - v^2)(n - p v^2)$, from which the proposition is evident. I conclude therefore that the results of Captain SABINE's observations, as far as they go, are in strict agreement with theory.

I now come to the comparison of the expression for the length of a meridian arc with those arcs which have been most accurately measured. The influence of local circumstances on these measures, it is well known, is greater than that on pendulum experiments; and the discordances in different measures are such that, in order to get a result of any exactness, we must confine ourselves to the longest arcs which have been measured with the greatest care. I have selected the following: 1. BOUGUER's arc in Peru: 2. LAMBTON's whole arc in India: 3. The French arc from Formentera to

Dunkirk: 4. The English arc from Dunnose to Clifton: 5. The Swedish arc from Mallorn to Pahtavara.

The Peruvian arc I have taken as recalculated by DELAMBRE in the Base du Systeme Metrique, Tome iii. p. 112, &c. I have supposed the toise of Peru equal to 1,065766 fathoms.

The Indian arc is taken as finally corrected by Colonel LAMBTON in the Philosophical Transactions for 1823.

In the estimation of the length of a part of the French arc, which Colonel LAMBTON has given in the Philosophical Transactions for 1818, and which Captain KATER has sanctioned by copying without any remark, there appears to be a serious error, arising from an unnecessary reduction for temperature. As I am sensible that I am now opposed to two Gentlemen whose assertion on such a point is almost decisive, I will state distinctly the reasons which have led me to this conclusion.

In the measure of the base at Melun (Base du Systeme Metrique, Tome ii. p. 44), the length of the base was reduced to the length which it would have had if the measuring rods had been used at the temperature of 13° of REAUMUR, or $16°\frac{1}{4}$ centigrade. The number of rods employed was four; but the rod or module No. 1 was compared with each of the others, and the base was expressed by the length of the module No. 1 at the temperature of $16°\frac{1}{4}$ centigrade.

From the logarithm set down in p. 698, it appears that this was the length used in the calculation of the triangles.

The module No. 1 was found at this temperature to be less than two toises (the toise being the length of an iron standard, called the toise of Peru, at the temperature of 13° of REAUMUR); but from an error in the zero point of the metallic thermometers there were some errors in the reduction of the base, which exactly corrected the difference (Tome iii. p. 136). The semi-module, therefore, in the expression for the arc of the meridian, is exactly equal to the toise of Peru at the temperature of $16°\frac{1}{4}$ centigrade.

The standard mètre at the temperature of 0° centigrade = 443,296 lines of the toise of Peru at $16°\frac{1}{4}$ centigrade (p. 139).

The standard mètre at the temperature of 0° centigrade was found by Captain KATER (Philosophical Transactions 1818) to be = 39,37079 inches of the English standard at the temperature of 62° Fahrenheit.

Consequently the semi-module in the expression for the arc of the meridian is = 1,065766 fathoms measured by the English standard at the temperature of 62° FAHRENHEIT; and since the English and Indian measures are estimated by this standard at the same temperature, no reduction is to be made for temperature. (The degree between Paris and Evaux is found thus to be 60822,5 fathoms; Col. LAMBTON has stated it to be 60779 fathoms). The length of the French arc given below, in fathoms, is therefore found by merely multiplying the number of semi-modules by 1,065766.

The English arc has the corrections which Captain KATER has given in the Phil. Trans. for 1821.

The Swedish arc is taken from SVANBERG's Exposition des Operations, &c. supposing their standard compared with the French standard at the temperature of 0° centigrade.

Thus the following table is formed.

Place.	L	L′	Length in Fathoms.
	° ′ ″	° ′ ″	
Peru . .	−0 2 31,22	3 4 31,9	188510
India . .	8 9 38,39	18 3 23,6	598630
France . .	38 39 56,11	51 2 9,2	751567
England .	50 37 5,27	53 27 29,89	172751
Sweden .	65 31 30,27	67 8 49,55	98870

In the application of the method of least squares, it must be observed, that the accuracy of the terrestrial measures can scarcely be questioned, and that the chance of errors in the determination of the extreme latitudes, arising either from faults of observation or from local attractions, is principally to be considered. This however amounts to the same as supposing an error in the length of the meridian arc. Assuming the form, M × number of seconds in $\overline{L'-L}$ + N (sin 2 L′ − sin 2 L) + P (sin 4 L′ − sin 4 L), the errors in the lengths are

$$-188510 + M \times 11223,1 + N \times ,1086 + P \times ,216$$
$$-598630 + M \times 35625,2 + N \times ,3084 + P \times ,412$$
$$-751567 + M \times 44533,1 + N \times ,0023 - P \times ,837$$
$$-172751 + M \times 10224,6 - N \times ,0241 - P \times ,175$$
$$-98870 + M \times 5839,3 - N \times ,0383 - P \times ,009$$

and the equations formed in the same manner as before are

$$0 = -59255232 + M \times 3516949,8 + N \times 11,8382 - P \times 22,016$$
$$0 = -198869 + M \times 11838,2 + N \times ,1090 + P \times ,153$$
$$0 = 37285 - M \times 2201,6 + N \times ,0153 + P \times ,093$$

from which M = 16,88164; N = −9358; P = 267; and the length of an arc of the meridian in fathoms =

$16{,}88164 \times$ nº. of seconds in $\overline{L' - L} - 9358 \times (\sin 2\,L' - \sin 2\,L) + 267 \times (\sin 4\,L' - \sin 4\,L)$.

The lengths of the several arcs found from this expression differ from those above by — 4, + 4, — 19, + 35, and + 63 fathoms respectively. The largest of these errors falls (as will generally be the case when the method of least squares is used) on that length in which the coefficients of M, N, and P, are smallest; but in a mountainous country, I conceive that an error of less than 4″ in the difference of latitudes is by no means inadmissible. And I am the more inclined to allow this error, because, upon applying the formula to the arc measured in Sweden by the French Academicians, the error, which is much greater than this, has a different sign.

I may here perhaps without impropriety, make some remarks on the credit which appears to be due to the French measure in Sweden. That measure has often been mentioned with contempt; but more particularly since the late measure by Svanberg. I have not however been able to discover any reason for this contempt, except the disagreement of its results from those of other operations. The last part of the process, it is well known, was the measure of the base; and the Academicians themselves were so much astonished at the result, that they immediately proceeded to verify every part of the measure, particularly the latitudes. And the general accuracy of the measure has been confirmed by the late measure. The triangles were admirably chosen; the length of the arc agrees within a few fathoms with that found by Svanberg; the latitude of the southern extremity at Tornea is precisely the same as that calculated by Svanberg from the observations at Mallorn; the latitude of Kittis, their northern station, unfortunately was not examined. The presumption, I think is, that though a part of the discrepancy may be attributed to errors of observation, yet a great part of it must be due to the irregularities of local attraction in so rugged a country (though SVANBERG appears to think this impossible), and that the new measure probably is not free from errors of the same kind. The measure at the Cape of Good Hope, conducted by the ablest astronomer of the age, has generally been thought inadmissible for the same reason.

Since $L' - L = \sin 1'' \times$ number of seconds in $L' - L$, we have for the length of a meridian arc $\frac{16,88164}{\sin 1''} \Big\{ L' - L - \frac{9358 \times \sin 1''}{16,88164} \times (\sin 2L' - \sin 2L) + \frac{267 \times \sin 1''}{16,88164} (\sin 4L' - \sin 4L) \Big\}$. Comparing this with the formula in (20), $\frac{3e}{4} - \frac{3e^2}{8} = \frac{9358 \times \sin 1''}{16,88164}$, and $\frac{15e^2}{64} - \frac{15A}{32} = \frac{267 \times \sin 1''}{16,88164}$. From these equations $e = ,003589 = \frac{1}{278,6}$; $A = -,000157$. The difference between the polar and equatorial axes is even greater than that assigned by Captain SABINE. But the most striking difference in the deductions is that the value of *A*, now found, has a negative sign; which would seem to indicate that the earth is protuberant at the latitude 45° above the ellipsoid, which has the same axes. And it does not appear that by any alteration of the values of A and e it is possible to reconcile the different observations. If we suppose the Indian and French arcs to be quite accurate, we shall find $e = ,003269 - A \times 2,139$: this evidently cannot be reconciled with the values of e and A deduced from the pendulum experiments.

On the whole I conceive, that we cannot assert that, on the figure of the earth, theory and observation are in perfect agreement. If the arcs which have been measured were no longer than that in Sweden, or even than that in England, I should not hesitate in attributing to errors of observation and local attraction a considerable part of the discrepancy. And had the theory been confined to terms of the first order, I should have thought it probable that terms of importance might be omitted in the second and higher orders. But it appears that with the improved theory, applied to the comparison of the Indian and French arcs, the shorter of which has an amplitude of nearly ten degrees, it is impossible to establish any agreement with the pendulum observations of Captain SABINE; and I know no other series which for extent and other advantages can be compared with those.

The measures of arcs of the meridian which have hitherto been made are, I imagine, insufficient for the determination of the figure of the earth. The arcs in India and France are the only ones in which the possible errors in latitude would not bear a sensible proportion to the effects of ellipticity. Captain SABINE has proposed to measure an arc in Spitzbergen. The shortness of the arc, and the mountainous character of the country, would make it almost useless. The desideratum at present is an extensive arc in a high latitude. We have two good arcs near the equator; of which the Indian, at least, is as long as can be desired. We have a still longer arc almost exactly bisected by the parallel of 45°. With an arc of equal length in the neighbourhood of the pole, we might determine the three terms in the expression for an arc of the meridian to great accuracy. This perhaps it will be impossible ever to execute.

There are some measures in which such extraordinary disturbances appear to have existed, that it seems desirable to examine and extend them: of these, the most remarkable is that made by LACAILLE at the Cape of Good Hope. And perhaps in speaking of the probable effects of local attraction, I may be permitted to allude to one of a different kind in England. This is the correction which it has been found necessary to apply to the longitudes of places in England, as deduced from the observations made at Beachy Head and Dunnose. For this investigation I prefer the method of DALBY, explained in the Philosophical Transactions for 1790 and 1795, to any other; as being unobjectionable on the ground of accuracy, and as applying to any surface in which the intersections of the normals at the two stations with the earth's axis, or with a line parallel to the earth's axis, are not very distant. The difference of longitude is then made to depend on this case of spherical trigonometry: given two sides (a, b) and the sum of the opposite angles $(A + B)$, to find the third angle (c). The formula is, $\tan. \frac{c}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cos. \frac{A+B}{2}$.

In this instance $a =$ colatitude of Dunnose $= 39° \; 22' \; 52'',69$; $b =$ colatitude of Beachy Head $= 39° \; 15' \; 36'',29$; $A + B =$ sum of observed azimuths $= 178° \; 52' \; 51''$; hence $C = 1° \; 26' \; 47'',87$. And if the errors in the observed quantities a, b, A, B, were $\delta a, \delta b, \delta A, \delta B$, the error in C would be

$$- \frac{\cos. \frac{a-b}{2} . \cos.^2 \frac{C}{2}}{\cos. \frac{a+b}{2} . \sin.^2 \frac{A+B}{2}} (\delta A + \delta B) + \frac{\cos. \frac{A+B}{2} . \cos.^2 \frac{C}{2}}{\cos.^2 \frac{a+b}{2}} (\sin. b . \delta a + \sin. a . \delta b) = -1,293 (\delta A + \delta B) + ,0103 . \delta a + ,0104 . \delta b.$$

Now the chronometer observations for the difference of longitude of Dover and Falmouth, according to Dr. TIARKS, (Phil. Trans. 1824) give by proportion the difference of longitude of Dunnose and Beachy Head 1° 27′ 4″,75. The discrepance then is nearly 17″; which (since a small error in *a* or *b* produces an almost insensible effect on C) implies an error of 13″ in the sum of the azimuths. This I should think is absolutely impossible. We must suppose then either that, in consequence of some local attraction, the apparent difference of longitudes was altered, or that some error has crept into the observations of Dr. TIARKS. The method of obtaining apparent time by corresponding altitudes, which was employed by Dr. TIARKS, is not the most accurate, nor perhaps the most proper, when the whole time in dispute is only five seconds. As Dr. TIARKS has not given the details of the observations, it is impossible to form any correct estimation of their accuracy. If however any local attraction have drawn aside the plummet at the eastern station towards the east, and that at the western station towards the west, the longitude would thus be made to appear smaller than it really is. The error produced in the longitude of either place would be $= \frac{\text{deviation}}{\text{cos. lat.}}$; and thus an error in longitude of 17″ would be accounted for by supposing the plummet at Beachy Head drawn 5″,4 to the east, and that at Dunnose 5″,4 to the west. Each of these is a larger deviation than we have reason to believe has taken place at any station in England; and it appears therefore desirable that the observations should be re-examined. This operation would not be difficult: the easiest method perhaps would be by rocket signals; and it is obvious, that the local causes which affected the difference of longitude found by observations of azimuth, would produce exactly the same effect by disturbing the levels of the transit or other instruments used to obtain the time. The immediate effect of such observations then, would be merely to confirm or to refute the conclusions deduced from the observations of azimuth: if the observations of Dr. TIARKS should be considered free from objection, they would serve to establish, or to destroy the belief, that local attraction may produce sensible disturbances in longitude.

G. B. AIRY.

Since the above was written, I have been favoured by Captain SABINE with a more detailed statement than I had before seen of the extent and probable circumstances of an arc in Spitzbergen. It appears that an arc might be measured extending nearly from latitude 76° 25′ to 80° 35′: the extremities of which would be on islands at a small distance from the main land of Spitzbergen. With the same formula as before, the error to be used in the method of least squares would be

$$-\text{measured length} + M \times 15000 - N \times ,1338 + P \times ,201.$$

Upon comparing this expression with those given before, it will be observed, that this is the only error in which the coefficient of N has a large negative value; and that from the preponderance at present of negative signs in the coefficients of P, an error with a positive coefficient of P would be very desirable. On these accounts then, an arc of that extent in that latitude would contribute much to our knowledge of the figure of the earth. But it is very likely that there would be sensible disturbances in the latitudes of the extreme stations. We cannot imagine a situation in which there is a greater probability that the difference of latitudes would be made too large, than when the latitudes are observed at two stations at the level of the sea, with high land between them. But this would depend much on the nature of the rocks. In Peru, for instance, though far more mountainous, it is probable that the disturbance was not great. If then, as Captain SABINE proposes, any preliminary survey be made, it would perhaps be proper to examine not only the circumstances which would affect the practical facility of the operation, but also those which might have an influence on the determination of the latitudes; and should these be found not very unfavourable, a meridian arc of four degrees measured with great care would be highly valuable.

ADDITIONS.

THE equations of Art. 15, it will easily be seen, are equally true, whether the interior of the mass be supposed to be fluid, or to consist of solid shells of different density, in which the radius of each separating surface is expressed by the formula $a \left\{ 1 + e . \overline{1 - \mu'^2} - \left(\frac{3e^2}{2} + A \right) \overline{. \mu'^2 - \mu'^4} \right\}$. And the expression for gravity in Art. 19, and all its consequences, hold equally on either supposition.

A theorem similar to CLAIRAUT'S may be shown to be true, to whatever order the investigations be extended. For upon using the value of R found in the investigation above, and upon carrying all the operations one step farther, it would be found that the equation of equilibrium could not be satisfied, in consequence of the introduction of terms multiplying μ'^6, which would not destroy each other. It would be necessary then to assume for R a value of the form

$$a\left\{1+e.\overline{1-\mu'^2}-\left(\frac{3e^2}{2}+A\right).\overline{\mu'^2-\mu'^4}-\left(\frac{e^3}{2}+B\right).\overline{\mu'^2-6\mu'^4+5\mu'^6}\right\}:$$

by which there would be introduced two new functions of *a, e,* &c. in the general expression for V, and in that for the force at any point; and one new function of a, e, &c. in the expression for the force at the surface. But as a power of μ' is introduced, which did not appear before, there would be found, by the process of Art. 14, one equation which was not found in the preceding approximation. By this equation the new function could be eliminated, and therefore the force at any point could be expressed in the same manner as in the preceding approximation, namely, by means of one function of (a) multiplying a quantity depending solely on the form of the external surface, and the latitude of the point on that surface. The same may be proved for every succeeding approximation; and thus we arrive at the following theorem: "If a heterogeneous fluid, the particles of which attract each other with accelerating forces inversely proportional to the squares of their distances, revolve round an axis; and if the proportion of the centrifugal force at the equator to the whole force there be given; the force at any point of the surface can be exactly expressed from a knowledge of the form of the surface and the position of that point, without any knowledge of the law of the internal density." This is likewise true if the interior be solid.

G. B. A.